A NEW INTEGRAL EQUATION FOR THE CALCULATION OF THE INTERNAL IMPEDANCE OF A CONDUCTOR

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Abstract

A new exact integral equation pertinent to the calculation of the skin effect induced internal impedance of a straight conductor, based on the correct boundary assumptions, is derived. Applied to a circular wire, it is shown that the exact internal impedance is persistently underestimated when employing classical techniques such as quasi-static approximation.

1 INTRODUCTION

As a consequence of the high-frequency spectral content of present-day digital signals, the study of the increase in internal impedance of conductors due to the skin effect has become a topic of increasing importance in computational electromagnetics. In this contribution we derive a new exact integral equation pertinent to the obtention of the internal impedance of a straight conductor, based on the correct boundary assumptions, and compare it with the classical quasi-static integral equation [1]. Applied to a circular wire, the comparison between the two integral equation formulations shows that there is a persistent underestimation of the internal impedance at microwave frequencies.

2 SKIN-EFFECT EQUATIONS

Consider, in the case of harmonic $e^{i\omega t}$ time dependence, a uniform straight conductor with an arbitrary cross-section $\mathcal{R}$ in the $x, y-$plane and suppose that the current density flows exclusively in the $z-$direction, i.e. $\mathbf{J} = \jmath(x, y)\,\mathbf{u}_z$. Without free internal current sources, the only current density is the one deriving from Ohm's law. Hence Maxwell's equations inside $\mathcal{R}$ can be written as

$$\begin{aligned} \nabla \times \mathbf{J} &= -i\omega\mu\sigma\,\mathbf{H} && (1) \\ \nabla \times \mathbf{H} &= \left(1 + \frac{i\omega\epsilon}{\sigma}\right)\mathbf{J} \approx \mathbf{J} && (2) \end{aligned}$$

where $\mu,\ \sigma$ and ϵ are the internal constitutive parameters. Since $\nabla\cdot\mathbf{J} = \partial\jmath/\partial z = 0$, we obtain the following Helmholtz equation for $\jmath$:

$$\nabla^2\jmath + k^2\jmath = 0 \tag{3}$$

where

$$k^2 = \mu\omega(\omega\epsilon - i\sigma) \tag{4}$$

Note that, for a good conductor, we may take $k^2 = -i\omega\mu\sigma$ since the term $\omega\epsilon/\sigma$ is very small up to microwave frequencies. E.g. for copper at 50 GHz we have $\omega\epsilon/\sigma = 4.65\,10^{-8} \ll 1$. The vector potential $\mathbf{A} = \mathsf{a}(x,y)\,\mathbf{u}_z$ outside $\mathcal{R}$ (free space) is given by

$$\mathsf{a}(\mathbf{r}) = -\mu_0 \int_{\mathcal{R}} g_0(\mathbf{r},\mathbf{r}')\jmath(\mathbf{r}')dS' \tag{5}$$

where the free space Green's function is

$$g_0(\mathbf{r},\mathbf{r}') = \frac{i}{4}\,H_0^{(2)}(k_0|\mathbf{r}-\mathbf{r}'|) \tag{6}$$

and $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the free space wavenumber. There are two boundary conditions at the interface of the conductor and free space : the continuity of the tangential magnetic field $\mathbf{H}_t$ and the continuity of the normal magnetic induction $\mathbf{B}_n$. If $\mathbf{H}$ in free space is expressed in terms of a and $\mathbf{H}$ inside $\mathcal{R}$ is expressed in terms of $\jmath$, and assuming that $\mu = \mu_0$, i.e. there is no magnetic contrast, we have the boundary condition

$$\nabla \times (\jmath\,\mathbf{u}_z + i\omega\sigma\mathsf{a}\,\mathbf{u}_z) = 0 \tag{7}$$

on $\partial\mathcal{R}$, the boundary of $\mathcal{R}$, implying [2] that

$$\jmath + i\omega\sigma\mathsf{a} = \text{constant} \quad \text{on} \quad \partial\mathcal{R} \tag{8}$$

Now, imposing a fixed potential difference $\mathcal{V}$ between $z = 0$ and $z = \ell$ amounts to requiring

$$\mathcal{V} = -\int_0^\ell \frac{\partial\psi}{\partial z}\,dz = \ell\left(\frac{\jmath}{\sigma} + i\omega\mathsf{a}\right) = \text{constant} \quad \text{on} \quad \partial\mathcal{R} \tag{9}$$

In (9), ψ stands for the scalar potential defined on the boundary as $-\frac{\partial\psi}{\partial z} = E_z + i\omega\mathsf{a}$, with of course $E_z = \jmath/\sigma$. Since the total current $\mathcal{I} = \int_{\mathcal{R}} \jmath\,d\,S$, we obtain the following formula for the internal impedance :

$$z_s = \frac{\mathcal{V}}{\mathcal{I}} = \ell\,\frac{\jmath + i\omega\sigma\mathsf{a}}{\sigma\int_{\mathcal{R}} \jmath\,d\,S} \tag{10}$$

In the DC case $\omega = 0$ we immediately obtain Pouillet's law $z_s = z_0 \equiv \ell/\sigma S$, since $\jmath$ is then constant over the entire cross-section $\mathcal{R}$. For $\omega > 0$ the current crowding skin-effect occurs, as will be discussed next.

3 A SURFACE INTEGRAL EQUATION

Putting $y_s = 1/z_s$ and $y_0 = 1/z_0$, we obtain the following formulation for the internal impedance. Let ϕ be the solution of the Helmholtz equation

$$\nabla^2\phi + k^2\phi = 0 \quad \text{in} \quad \mathcal{R} \tag{11}$$

with boundary condition

$$\phi + i\omega\sigma\mathsf{a} = 1 \quad \text{on} \quad \partial\mathcal{R} \tag{12}$$

Then the normalized internal admittance $\tilde{y}_s$ and normalized internal impedance $\tilde{z}_s$ are given by

$$\tilde{y}_s = \frac{1}{\tilde{z}_s} = \frac{y_s}{y_0} = \frac{z_0}{z_s} = \frac{1}{S}\int_{\mathcal{R}} \phi\,d\,S \tag{13}$$

Now putting $\phi = 1 - i\omega\sigma \mathsf{a} + v$, it is clear that $v = 0$ on the boundary, while satisfying

$$\nabla^2 v + i\omega\sigma k_0^2 \mathsf{a} + (k^2 + i\omega\mu\sigma)\phi = 0 \tag{14}$$

inside $\mathcal{R}$. This follows from the fact that

$$\nabla^2 \mathsf{a} + k_0^2 \mathsf{a} = -\mu\phi \tag{15}$$

since we have

$$\mathsf{a}(\mathbf{r}) = -\mu \int_{\mathcal{R}} g_0(\mathbf{r}, \mathbf{r}')\phi(\mathbf{r}')dS' \tag{16}$$

With $k^2 = -i\omega\mu\sigma$, equation (14) simplifies to

$$\nabla^2 v + i\omega\sigma k_0^2 \mathsf{a} = 0 \tag{17}$$

Equation (17) can be solved by means of the Dirichlet kernel

$$\mathcal{D}(\mathbf{r}, \mathbf{r}') = -\sum_n \frac{1}{\lambda_n} u_n(\mathbf{r}) u_n(\mathbf{r}') \tag{18}$$

where $\lambda_n,\ u_n(\mathbf{r})$ are the Dirichlet eigenvalues and orthonormalized eigenfunctions for $\mathcal{R}$, yielding

$$v(\mathbf{r}) = -i\omega\sigma k_0^2 \int_{\mathcal{R}} \mathcal{D}(\mathbf{r}, \mathbf{r}')\mathsf{a}(\mathbf{r}')\, dS' \tag{19}$$

The equation $\phi = 1 - i\omega\sigma \mathsf{a} + v$ therefore represents a surface integral equation, which can be written in an easily understood fashion as

$$\phi + k^2 [g_0]\, \phi + k^2 k_0^2 [\mathcal{D}]\, [g_0]\, \phi = 1 \tag{20}$$

This is the integral equation we are looking for. For an electrically small conductor, with its largest linear dimension much smaller than the free space wavelength $2\pi/k_0$, we may tentatively take $k_0 = 0$ in (20), yielding the well-known quasi-static surface integral equation [1]

$$\phi + k^2 [g_{00}]\, \phi = 1 \tag{21}$$

where $[g_{00}]$ stands for the logarithmic kernel

$$g_{00}(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi}\, \ln |\mathbf{r} - \mathbf{r}'| \tag{22}$$

Of course, at microwave frequencies, it would be interesting to evaluate the difference between the internal impedance calculated by means of the exact integral equation (20) versus the internal impedance calculated by means of the approximate integral equation (21).

This can be done analytically in the case of a circular wire of radius a [3] where we obtain

$$z_s = Z_1(f) \equiv z_0 \left(\frac{kaJ_0(ka)}{2J_1(ka)} - \frac{1}{2} i\omega\sigma\mu a^2 \ln a \right) \tag{23}$$

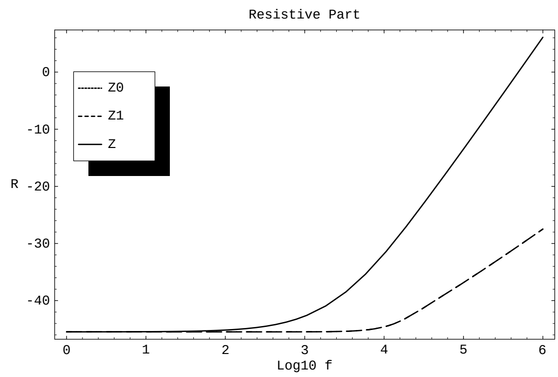
when calculated by means of the approximate equation (21). If we discard the term with the logarithm, we obtain the classical formula [4]

$$z_s = Z_0(f) \equiv z_0\, \frac{kaJ_0(ka)}{2J_1(ka)} \tag{24}$$

When utilizing the exact integral equation (20), we obtain, after some tedious algebraic manipulations

$$z_s = Z(f) \equiv \frac{kaz_0}{2J_1(ka)} \left\{ J_0(ka) + \frac{1}{2}\,\pi\omega\sigma\mu H_0^{(2)}(k_0a)\,\frac{kaJ_1(ka)J_0(k_0a) - k_0aJ_1(k_0a)J_0(ka)}{k^2 - k_0^2} \right\} \tag{25}$$

In Figure 1 we show the resistive part in dB of $Z_0(f), Z_1(f)$ and $Z(f)$ for a 1 m long copper wire ($\sigma = 5.8 \times 10^7\,\mathrm{S\,m^{-1}}, \epsilon = \epsilon_0$) of diameter 2 mm as a function of frequency up to 1 GHz. It is seen that the difference between $R(f)$ and $R_0(f) = R_1(f)$ is striking, in the sense that the resistance is underestimated in general.



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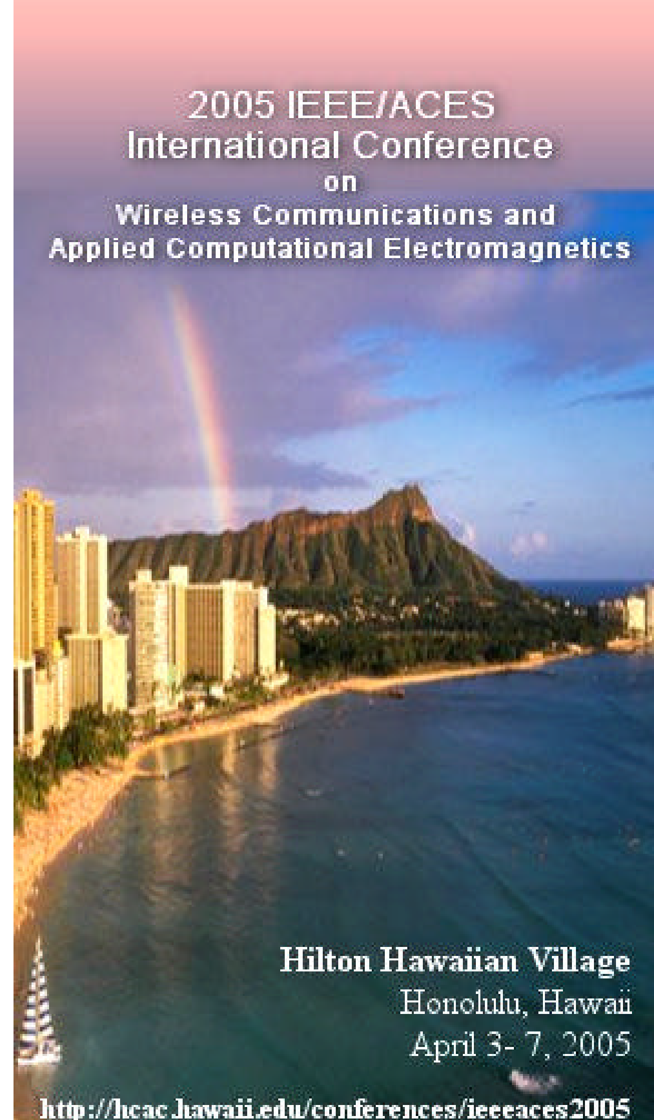

April 4	1:20-5:00 PM	South Pacific 4
8	**Integral Equation Methods and Applications**	

1:20	An Integral Equation Method for the Scattering from Multiple Multilayered cylinders *Fad Seydou, University of Oulu, Finland*
1:40	A New Integral Equation for the Calculation of the Internal Impedance of a Conductor *Luc Knockaert, Ghent University, Belgium*
2:00	The Effect of Integration Accuracy on the MoM VIE Solution for Dielectric Resonators *Shashank Kulkarni, Sergey Makarov, WorchesterWorcester Polytechnic Institute, USA*
2:20	Bistatic Scattering from a PEMC (Perfect Electromagnetic Conducting) Sphere: Surface Integral Equation Approach *Ari Sihvola, Pasi Yla-Oijala, Ismo V. Lindell, Helsinki University of Technology, Finland*
2:40	2D MFIE Solution Improvement by Regularization *Clayton P. Davis, Karl F. Warnick, Brigham Young University, USA*
3:00	**Coffee Break**
3:20	Combined-Field Solution of Composite Geometries Involving Open and Closed Conducting Surfaces *Ozgur Ergul, Levent Gurel, Bilkent University, Turkey*
3:40	Formulation of surface integral equations for metallic, dielectric and composite objects *Pasi Ylä-Oijala, Matti Taskinen, Helsinki University of Technology, Finland*
4:00	A Simple Extrapolation Method Based on Current for Rapid Frequency and Angle Sweep in Far-Field Calculation of an Integral Equation Algorithm *Cai-Cheng Lu, University of Kentucky, USA*
4:20	Fast Construction of Wavelet-Based Moment Matrices in Solving Thin-Wire Electric Field Integral Equations *Mr. Amir Geranmayeh, Prof. Rouzbeh Moini, Prof. S. H. Hesam Sadeghi, Amirkabir University of Technology, Iran*
4:40	Eddy currents in a gradient coil, modeled by rings and patches *J.M.B. Kroot, S.J.L van Eijndhoven, A.A.F. van de Ven, Eindhoven University of Technology, Netherlands*